Journal of Pure and Applied Algebra 38 (1985) 187-211
North-Holland

COHOMOLOGY OF SMALL CATEGORIES

Hans-Joachim BAUES and Günther WIRSCHING
Math. Institut der Universität Bonn, Wegelerstrasse 10, 53 Bonn, F.R.G.

Communicated by J.D. Stasheff
Received 25 September 1984
Revised 7 February 1985

Dedicated to Jan-Erik Roos on his 50-th birthday

In this paper we introduce and study the cohomology of a small category with coefficients in a natural system. This generalizes the known concepts of Watts [23] (resp. of Mitchell [17]) which use modules (resp. bimodules) as coefficients. We were led to consider natural systems since they arise in numerous examples of linear extensions of categories; in Section 3 four examples are discussed explicitly which indicate deep connection with algebraic and topological problems:

(1) The category of $\mathbb{Z}/p^2$-modules, p prime.

(2) The homotopy category of Moore spaces in degree n, $n \geq 2$.

(3) The category of group rings of cyclic groups.

(4) The homotopy category of Eilenberg-MacLane fibrations.

We prove the following results on the cohomology with coefficients in a natural system:

(5) An equivalence of small categories induces an isomorphism in cohomology.

(6) Linear extensions of categories are classified by the second cohomology group H^2.

(7) The group H^1 can be described in terms of derivations.

(8) Free categories have cohomological dimension ≤ 1, and category of fractions preserve dimension one.

(9) A double cochain complex associated to a cover yields a method of computation for the cohomology; two examples are given.

The results (7) and (8) correspond to known properties of the Hochschild-Mitchell cohomology, see [7] and [17].

In the final section we discuss the various notions of cohomology of small categories, and we show that all these can be described in terms of Ext functors studied in the classical paper [11] of Grothendieck.

Notation

We use the following notations: A boldface letter like **C** denotes a category, Ob(**C**) and Mor(**C**) are the classes of objects and morphisms respectively. We identify an object A with its identity $1_A = 1$. The set of morphisms $A \to B$ is $\mathbf{C}(A, B)$, and the group of automorphisms of A is $\mathrm{Aut}_{\mathbf{C}}(A)$.

1. Cohomology of a small category

An early approach to the cohomology of small categories is due to Jan Erik Roos [22] in his classical result on the derived $\varprojlim^{(n)}$ of the $\varprojlim$ functor. As pointed out by Quillen [21] the singular cohomology of the classifying space of a small category is an example of $\varprojlim^{(n)}$. On the the other hand, Mitchell [17] introduces a cohomology by imitating as closely as possible the classical ring theory on the level of categories. This Hochschild–Mitchell cohomology uses bimodules as coefficients, while [21] and [23] use modules. The approach here generalizes these two concepts by taking 'natural systems' as coefficients which are more adapted to categories. Indeed, a module (resp. a bimodule) associates an abelian group to an object (resp. to a pair of objects), while a natural system associates an abelian group to each morphism.

Let **C** be a category. The *category of factorizations* in **C**, denoted by $F\mathbf{C}$, is given as follows: Objects are the morphisms $f, g, \ldots$ in **C** and morphism $f \to g$ are pairs (α, β) for which

$$(1.1) \qquad \begin{array}{ccc} B & \xrightarrow{\ \alpha\ } & B' \\ \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle g} \\ A & \xleftarrow{\ \beta\ } & A' \end{array}$$

commutes in **C**. Composition is defined by $(\alpha', \beta')(\alpha, \beta) = (\alpha'\alpha, \beta\beta')$. We clearly have $(\alpha, \beta) = (\alpha, 1)(1, \beta) = (1, \beta)(\alpha, 1)$ where 1 denotes the identity. This is the 'twisted arrow category' in [27].

A *natural system* (of abelian groups) on **C** is a functor,

$$(1.2) \qquad D : F\mathbf{C} \to \mathbf{Ab},$$

from the category of factorizations to the category of abelian groups. The functor D carries the object f to $D_f = D(f)$ and carries the morphism $(\alpha, \beta) : f \to g$ above to the induced homomorphism

$$(1.3) \qquad D(\alpha, \beta) = \alpha_* \beta^* : D_f \to D_{\alpha f \beta} = D_g$$

where $D(\alpha, 1) = \alpha_*$ and $D(1, \beta) = \beta^*$.

(1.4) **Definition.** Let **C** be a small category. We define the *cohomology* $H^n(\mathbf{C}, D)$ of **C** *with coefficients in the natural system* D by the cohomology of the following cochain complex $\{F^n, \delta\}$. The n-th cochain group $F^n = F^n(\mathbf{C}, D)$ is the abelian group of all functions

(a) $$f : N_n(\mathbf{C}) \to \dot{\bigcup_{g \in \mathrm{Mor}(\mathbf{C})}} D_g \text{ with } f(\lambda_1, \dots, \lambda_n) \in D_{\lambda_1 \circ \cdots \circ \lambda_n}.$$

Here $N_n(\mathbf{C})$ is the set of sequences $(\lambda_1, \dots, \lambda_n)$ of n composable morphisms

$$A_0 \xleftarrow{\lambda_1} A_1 \leftarrow \cdots \xleftarrow{\lambda_n} A_n$$

in $\mathbf{C}$ (which are the n-simplices of the *nerve* of $\mathbf{C}$). For $n = 0$ let $N_0(\mathbf{C}) = \mathrm{Ob}(\mathbf{C})$ be the set of objects in $\mathbf{C}$ and let $F^0(\mathbf{C}, D)$ be the set of all functions

(a)′ $$f : \mathrm{Ob}(\mathbf{C}) \to \dot{\bigcup_{A \in \mathrm{Ob}(\mathbf{C})}} D_A$$

with $f(A) \in D_A = D(1_A)$. Addition in F^n is given by adding pointwise in the abelian groups D_f. The coboundary

(b) $$\delta : F^{n-1} \to F^n$$

is defined by the formula ($n > 1$):

(c) $$\begin{aligned}(\delta f)(\lambda_1, \dots, \lambda_n) &= \lambda_{1*} f(\lambda_2, \dots, \lambda_n) \\ &+ \sum_{i=1}^{n-1} (-1)^i f(\lambda_1, \dots, \lambda_i \lambda_{i+1}, \dots, \lambda_n + (-1)^n \lambda_n^* F(\lambda_1, \dots, \lambda_{n-1}).\end{aligned}$$

For $n = 1$ the coboundary δ in (b) is given by

(c)′ $$(\delta f)(\lambda) = \lambda_* f(A) - \lambda^* f(B) \quad \text{for } (\lambda : A \to B) \in N_1(\mathbf{C}).$$

One can check by (1.3) that $\delta f \in F^n$ for $f \in F^{n-1}$ and that $\delta\delta = 0$.

We now describe the natural properties of the cohomology. To this end we introduce the *category* **Nat** *of all natural systems*. Objects are pairs $(\mathbf{C}, D)$ where D is a natural system of the small category $\mathbf{C}$. Morphisms are pairs

(1.5) $$(\phi^{\mathrm{op}}, \tau) : (\mathbf{C}, D) \to (\mathbf{C}', D')$$

where $\phi : \mathbf{C}' \to \mathbf{C}$ is a functor and where $\tau : \phi^* D \to D'$ is a natural transformation of functors. Here $\phi^* D : F\mathbf{C}' \to \mathbf{Ab}$ is given by

(1.6) $$(\Phi^* D)_f = D_{\phi f} \quad \text{for } f \in \mathrm{Mor}(\mathbf{C}')$$

and $\alpha_* = \phi(\alpha)_*$, $\beta^* = \phi(\beta)^*$. A natural transformation $t : D \to \tilde{D}$ yields as well the natural transformation

(1.7) $$\phi^* t : \phi^* D \to \phi^* \tilde{D}.$$

Now morphisms in **Nat** are composed by the formula

(1.8) $$(\psi^{\mathrm{op}}, \sigma)(\phi^{\mathrm{op}}, \tau) = ((\phi\psi)^{\mathrm{op}}, \sigma \circ \psi^* \tau).$$

The cohomology introduced above is a *functor*,

(1.9) $$H^n : \mathbf{Nat} \to \mathbf{Ab} \qquad (n \in \mathbb{Z}),$$

which carries the morphism $(\phi^{\mathrm{op}}, \tau)$ of (1.5) to the induced homomorphism

(1.10) $$\phi^* \tau_* : H^n(\mathbf{C}, D) \to H^n(\mathbf{C}', D')$$

given on cochains $f \in F^n$ by $(\phi^* \tau_* f)(\lambda'_1, \ldots, \lambda'_n) = \tau_f \circ f(\phi\lambda'_1, \ldots, \phi\lambda'_n)$. We have $(\phi^{\mathrm{op}}, \tau) = (\phi^{\mathrm{op}}, 1)(1, \tau) = (1, \phi^*\tau)(\phi^{\mathrm{op}}, 1)$ and we write $\phi^* = (\phi^{\mathrm{op}}, 1)_*$ and $(1, \tau)_* = \tau_*$.

(1.11) **Theorem.** *Suppose $\phi : \mathbf{C}' \to \mathbf{C}$ is an equivalence of small categories. Then ϕ induces an isomorphism*

$$\phi^* : H^n(\mathbf{C}, D) \cong H^n(\mathbf{C}', \phi^* D)$$

for all natural systems D on $\mathbf{C}$, $n \in \mathbb{Z}$.

For the proof of this result we consider first a natural equivalence

$$t : \phi \cong \psi, \qquad \phi, \psi : \mathbf{C}' \to \mathbf{C}$$

which induces an isomorphism of natural systems

(1.12) $$\tilde{t} : \phi^* D \cong \psi^* D \quad \text{with } \tilde{t} = t_*(t^{-1})^* : D_{\phi f} \cong D_{\psi f}.$$

Here we have $\psi f = t(\phi f)t^{-1}$ since t is a natural equivalence.

(1.13) **Lemma.** $\tilde{t}_* \phi^* = \psi^*$ *on* $H^n(\mathbf{C}, D)$.

(1.14) **Proof of Theorem (1.11).** Let $\phi' : \mathbf{C} \to \mathbf{C}'$ be a functor and let

$$t : \phi'\phi \cong 1, \qquad \tau : \phi\phi' \cong 1$$

be equivalences. Then by (1.13) we have

$$\tilde{t}_*(\phi'\phi)^* = 1^* = 1 \quad \text{and} \quad \tilde{\tau}_*(\phi\phi')^* = 1^* = 1.$$

Here $\tilde{t}_*$ and $\tilde{\tau}_*$ are isomorphisms and therefore ϕ^* is an isomorphism. □

(1.15) **Proof of Lemma (1.13).** We construct a chain homotopy h for the diagram of cochain maps

(1) $$\begin{array}{ccc} & & F^*(\mathbf{C}', \phi^* D) \\ & \overset{\phi^*}{\nearrow} & \\ F^*(\mathbf{C}, D) & & \big\downarrow \tilde{t}_* \\ & \underset{\psi^*}{\searrow} & \\ & & F^*(\mathbf{C}', \psi^* D), \end{array}$$

(2) $$\tilde{t}_* \phi^* - \psi^* = \delta h + h\delta \quad \text{with } h : F^{n+1}(\mathbf{C}, D) \to F^n(\mathbf{C}', \psi^* D).$$

Here h is given by the following formula

$$(3) \qquad (hf)(\lambda_1', \ldots, \lambda_n') = (t^*)^{-1} \sum_{i=0}^{n} (-1)^i f(\psi\lambda_1, \ldots, \psi\lambda_i, t, \phi\lambda_{i+1}, \ldots, \phi\lambda_n)$$

The terms in the alternating sum correspond to paths in the commutative diagram

$$(4) \qquad \begin{array}{ccccccccc} * & \xleftarrow{\phi\lambda_1} & * & \xleftarrow{\phi\lambda_2} & \cdots & * & \xleftarrow{\phi\lambda_n} & * \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \cdots & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} \\ * & \xleftarrow[\psi\lambda_1]{} & * & \xleftarrow[\psi\lambda_2]{} & \cdots & * & \xleftarrow[\psi\lambda_n]{} & * \end{array}$$

A somewhat tedious but straightforward calculation shows that formula (2) is satisfied for h. □

There are various special cases of natural systems which we obtain by the functors:

$$(1.16) \qquad F\mathbf{C} \xrightarrow{\pi} \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \xrightarrow{p} \mathbf{C} \xrightarrow{q} \pi\mathbf{C} \xrightarrow{0} *.$$

Here π and p are the obvious forgetful functors and q is the localization functor for the *fundamental groupoid*:

$$(1.17) \qquad \pi\mathbf{C} = (\mathrm{Mor}\,\mathbf{C})^{-1}\mathbf{C}, \qquad \text{see [10].}$$

Moreover $*$ in (1.16) is the trivial category consisting of one object and one morphism and 0 is the trivial functor. Using the functors in (1.16) we get special natural systems on **C** by pulling back functors in **Fun(K, Ab)** where **K** is one of the categories in (1.16). Such functors are denoted as follows:

(1.18) **Definition.** M is a **C**-*bimodule* if $M \in \mathbf{Fun}(\mathbf{C}^{\mathrm{op}} \times \mathbf{C}, \mathbf{Ab})$.

F is a **C**-*module* if $F \in \mathbf{Fun}(\mathbf{C}, \mathbf{Ab})$.

L is a *local system* on **C** if $L \in \mathbf{Fun}(\pi\mathbf{C}, \mathbf{Ab})$.

A is a *trivial system* on **C** if A is an abelian group or equivalently if $A \in \mathbf{Fun}(*, \mathbf{Ab})$.

Clearly we define the *cohomology of* **C** *with coefficients in* M, F, L and A respectively by the groups

$$(1) \qquad H^n(\mathbf{C}, M) = H^n(\mathbf{C}, \pi^* M),$$

$$(2) \qquad H^n(\mathbf{C}, F) = H^n(\mathbf{C}, \pi^* p^* F),$$

$$(3) \qquad H^n(\mathbf{C}, L) = H^n(\mathbf{C}, \pi^* p^* q^* L),$$

$$(4) \qquad H^n(\mathbf{C}, A) = H^n(\mathbf{C}, \pi^* p^* q^* 0^* A).$$

(1.19) **Remark.** As we will show in Section 8, the cohomology (1) can be identified with the cohomology introduced by Hochschild and Mitchell, see [17] and [7]. Moreover the cohomology (2) is the one used by Roos [22], Quillen [21] and Grothendieck (see [15] for the definition of topos cohomology). Next the cohomologies (3) and (4) are the usual singular cohomologies of the classifying space $B\mathbf{C}$ with local coefficients and with coefficients in an abelian group respectively.

(1.20) **Remark.** The cohomology $H^n(\mathbf{C}, D)$ with coefficients in a natural system as well generalizes canonically the *cohomology of a group G with coefficients in a right G-module A*: We denote the action of $\xi \in G$ on $a \in A$ by a^ξ. Consider the *category* $\mathbf{G}$ with a single object and with $\mathrm{Mor}(\mathbf{G}) = G$. Then one has a *natural system* D^A on $\mathbf{G}$ by

$$(D^A)_f = A \quad \text{for } f \in G \quad \text{and } \alpha_* = 1,\ \beta^*(a) = a^\beta \quad \text{for } a \in A,\ \alpha, \beta \in G.$$

Now one can check by the usual definition of $H^n(G, A)$ that (1.4) yields the equation

$$H^n(G, A) = H^n(\mathbf{G}, D^A).$$

Compare for example [6], [14], [16].

2. Linear extensions of categories and H^2

An extension of a group G by a G-module A is a short exact sequence of groups

$$(2.1) \qquad 0 \to A \xrightarrow{i} E \xrightarrow{p} G \to 0$$

where i is compatible with the action of G, namely $i(a^\xi) = x^{-1}(ia)x$ for $x \in p^{-1}(\xi)$. Two such extensions E and E' are equivalent if there is an isomorphism $\varepsilon : E \cong E'$ of groups with $p'\varepsilon = p$ and $\varepsilon i = i'$. It is wellknown that the equivalence classes of extensions are classified by the cohomology $H^2(G, A)$ in (1.20).

We now consider linear extensions of a small category $\mathbf{C}$ by a natural system D and we show that the equivalence classes of such extensions are equally classified by the cohomology $H^2(\mathbf{C}, D)$ defined in Section 1.

(2.2) **Definition.** We say that

$$D + \to \mathbf{E} \xrightarrow{p} \mathbf{C}$$

is a *linear extension* if (a), (b) and (c) hold:

(a) $\mathbf{E}$ and $\mathbf{C}$ have the same objects and p is a full functor which is the identity on objects.

(b) For each morphism $f : A \to B$ in $\mathbf{C}$ the abelian group D_f acts transitively and effectively on the subset $p^{-1}(f)$ of morphisms in $\mathbf{E}$. We write $f_0 + \alpha$ for the action of $\alpha \in D_f$ on $f_0 \in p^{-1}(f)$.

(c) The action satisfies the *linear distributivity law:*

$$(f_0+\alpha)(g_0+\beta)=f_0g_0+f_*\beta+g^*\alpha.$$

Two linear extensions $\mathbf{E}$ and $\mathbf{E}'$ are *equivalent* if there is an isomorphism $\varepsilon:\mathbf{E}\cong\mathbf{E}'$ of categories with $p'\varepsilon=p$ and with $\varepsilon(f_0+\alpha)=\varepsilon(f_0)+\alpha$ for $f_0\in\mathrm{Mor}(\mathbf{E})$, $\alpha\in D_{pf_0}$.

The extension $\mathbf{E}$ is a *split extension* if there is a functor $s:\mathbf{C}\to\mathbf{E}$ with $ps=1$.

(2.3) **Theorem** (Classification). *Let D be a natural system on a small category $\mathbf{C}$ and let $M(\mathbf{C},D)$ be the set of equivalence classes of linear extensions of $\mathbf{C}$ by D. Then there is a canonical bijection*

$$\psi:M(\mathbf{C},D)\cong H^2(\mathbf{C},D)$$

which maps the split extension to the zero element in the cohomology group $\mathrm{H}^2(\mathbf{C},D)$.

(2.4) **Example.** Let G be a group and let A be a right G-module. For the natural system D^A on $\mathbf{G}$ in (1.20) the set $M(\mathbf{G},D^A)$ can be identified easily with the set $E(G,A)$ of all equivalence classes of extensions in (2.1). Therefore (2.3) and (1.20) yield the result:

$$E(G,A)=M(\mathbf{G},D^A)=H^2(\mathbf{G},D^A)=H^2(G,A).$$

This, in fact, is the wellknown classification of group extensions, see for example [14], [16].

Proof of Theorem (2.3). Let $p:\mathbf{E}\to\mathbf{C}$ be a linear extension by D. Since p is surjective on morphisms there extists a function

(1) $\qquad s:\mathrm{Mor}(\mathbf{C})\to\mathrm{Mor}(\mathbf{E})$

with $ps=1$. If we have two such functions s and s' the condition $ps=1=ps'$ implies that there is a unique element

(2) $\qquad d\in F^1(\mathbf{C},D)\quad$ with $s'(f)=s(f)+d(f),\quad f\in\mathrm{Mor}(\mathbf{C})$.

Moreover, each $d\in F^1(\mathbf{C},D)$ gives us by $(s+d)(f)=s(f)+d(f)$ a function $s+d:\mathrm{Mor}(\mathbf{C})\to\mathrm{Mor}(\mathbf{E})$ with $p(s+d)=ps=1$.

For a $(y,x)\in N_2(\mathbf{C})$ the formula

(3) $\qquad s(yx)=s(y)s(x)+\Delta_s(y,x)$

determines the element

(4) $\qquad \Delta_s\in F^2(\mathbf{C},D)$.

This element measures the deviation of s from being a functor. If s is a splitting, then $\Delta_s=0$. We now define the function Ψ in (2.3) by

(5) $\qquad \Psi\{\mathbf{E}\}=\{\Delta_s)$.

Here $\{\mathbf{E}\} \in M(\mathbf{C}, D)$ is the equivalence class of the extension $\mathbf{E}$ and $\{\Delta_s\} \in H^2(\mathbf{C}, D)$ is the cohomology class represented by the cocycle Δ_s in (4) where s is chosen as in (1). First we have to check the cocycle condition for Δ_s: We compute

$$s((zy)x) = s(z)s(y)s(x) + x^*\Delta_s(z, y) + \Delta_s(zy, x),$$

$$s(z(yx)) = s(z)s(y)s(x) + z_*\Delta_s(y, x) + \Delta_s(z, yx).$$

Therefore associativity of composition implies

(6)
$$\begin{aligned} 0 &= z_*\Delta_s(y, x) - \Delta_s(zy, x) + \Delta_s(z, yx) - x^*\Delta_s(z, y) \\ &= (\delta\Delta_s)(z, y, x), \qquad \text{see (1.4)(c)}. \end{aligned}$$

Moreover the cohomology class $\{\Delta_s\}$ does not depend on the choice of s: We compute

$$(s+d)(yx) = s(y)s(x) + x^*d(y) + y_*d(x) + \Delta_{s+d}(y, x).$$

Therefore we have by (3)

(7)
$$\begin{aligned} \Delta_s(y, x) - \Delta_{s+d}(y, x) &= y_*d(x) - d(yx) + x^*d(y) \\ &= (\delta d)(y, x), \qquad \text{see (1.4)(c)}. \end{aligned}$$

In addition, we see that for an equivalence ε we have

(8) $$\Delta_{\varepsilon s} = \Delta_s.$$

By (6), (7) and (8) the function ψ in (5) is well defined. The function Ψ is surjective by the following construction: Let $\Delta \in F^2(\mathbf{C}, D)$, $\delta\Delta = 0$. We get an extension

(9) $$p_\Delta : \mathbf{E}_\Delta \to \mathbf{C} \quad \text{with } \Psi\{\mathbf{E}_\Delta\} = \{\Delta\}.$$

The morphisms in $\mathbf{E}_\Delta$ are the pairs (f, α) with $f \in \mathrm{Mor}(\mathbf{C})$, $\alpha \in D_f$. The composition in $\mathbf{E}_\Delta$ is defined by

(10) $$(g, \beta)(f, \alpha) = (gf, \Delta(g, f) + g_*\alpha + f^*\beta).$$

The action of D on $\mathbf{E}_\Delta$ is defined by $(f, \alpha) + \alpha' = (f, \alpha + \alpha')$, $\alpha' \in D_f$.
Since we have an equivalence

(11) $$\mathbf{E}_{\Delta_s} \xrightarrow{\varepsilon} \mathbf{E} \quad \text{with } \varepsilon(f, \alpha) = s(f) + \alpha$$

we see that Ψ is also injective. □

(2.5) **Remark.** For a linear extension

(1) $$D+ \to \mathbf{E} \to \mathbf{C}$$

the corresponding cohomology class $\Psi\{\mathbf{E}\} \in H^2(\mathbf{C}, D)$ has the following *universal property* with respect to the groups of automorphisms in $\mathbf{E}$: For an object A in $\mathbf{E}$ the extension (1) yields the group extension

(2) $$0 \to \bar{A} \to \mathrm{Aut}_{\mathbf{E}}(A) \to \mathrm{Aut}_{\mathbf{C}}(A) \to 0$$

by restriction. Here $\alpha \in \mathrm{Aut}_{\mathbf{C}}(A)$ acts on $x \in \bar{A} = D(1_A)$ by $x^{\alpha} = (\alpha^{-1})_* \alpha^*(x)$. The cohomology class corresponding to the extension (2) is given by the image of the class $\Psi\{\mathbf{E}\}$ under the homomorphism

(3) $$H^2(\mathbf{C}, D) \xrightarrow{t_* i^*} H^2(\mathrm{Aut}_{\mathbf{C}}(A), \bar{A}).$$

Here i is the inclusion functor $\mathrm{Aut}_{\mathbf{C}}(A) \hookrightarrow \mathbf{C}$ and $t: i^*D \to D^{\bar{A}}$ is the isomorphism of natural systems, see (1.20), with

(4) $$t = (\alpha^{-1})_* : D_{i\alpha} \to D(1_A) = \bar{A}.$$

3. Algebraic and topological examples of linear extensions

(3.1) *The category of $\mathbb{Z}/p^2$-modules (p prime)*

Let R be a commutative ring and let $\mathbf{M}_R$ be the (small) category of finitely generated free R-modules: Objects are

(1) $$R^n = R \oplus \cdots \oplus R, \quad n \text{ summands}, \ n \geq 1.$$

and morphisms $R^n \to R^m$ are $m \times n$-matrices (α_{ij}) over R, and composition is multiplication of matrices. We have the canonical $\mathbf{M}_R$-bimodule

(2) $$\mathrm{Hom}: \mathbf{M}_R^{\mathrm{op}} \times \mathbf{M}_R \to \mathbf{Ab}$$

which carries (R^n, R^m) to the abelian group $\mathrm{Hom}(R^n, R^m) = M^{m,n}(R)$ of $m \times n$-matrices.

For any prime p there is the canonical linear extension of categories

(3) $$\mathrm{Hom} + \to \mathbf{M}_{\mathbb{Z}/p^2} \xrightarrow{q} M_{\mathbb{Z}/p}.$$

Here q is reduction mod p and the matrix $\beta = (\beta_{ij}) \bmod p$ acts on the matrix $\alpha = (\alpha_{ij}) \bmod p^2$ by the formula

(4) $$\alpha + \beta = (\alpha_{ij} + p \cdot \beta_{ij}).$$

It is an easy exercise to show that (3) is a welldefined linear extension of categories. By a result of Werner Meyer (Max-Planck-Institut für Mathematik in Bonn) the extension (3) is not split. Therefore by (2.3) it represents a nontrivial cohomology class, a generator in fact, of

(5) $$H^2(\mathbf{M}_{\mathbb{Z}/p}, \mathrm{Hom}) = \mathbb{Z}/p.$$

If restricted to the group of automorphisms the extension in (3) yields elements in

(6) $$H^2(\mathrm{GL}_n(\mathbb{Z}/p), M^{n,n}(\mathbb{Z}/p))$$

which are nontrivial for $(n-1)(p-1) \geq 2$. These elements actually turned out to be of importance in computations on the cohomology of the general linear groups $GL_n(\mathbb{Z}/p)$, [9].

(3.2) *The homotopy category of Moore spaces in degree n, $n \geq 2$*

Let $\mathbf{Ab}_0$ be the small category of finitely generated abelian groups. For each $A \in \mathbf{Ab}_0$ we choose a Moore space $M(A,n)$. This is a simply connected CW-complex with a single nontrivial homology group in degree n isomorphic to A. Let $\mathbf{Moore}^n$ be the full homotopy category of such Moore spaces. Then there is a canonical linear extension of categories

(1) $\qquad E^n + \rightarrow \mathbf{Moore}^n \xrightarrow{H_n} \mathbf{Ab}_0.$

Here H_n is the n-th homology functor and E^n is the following bifunctor on $\mathbf{Ab}_0$:

(2) $\qquad E^n(A,B) = \mathrm{Ext}^1_{\mathbb{Z}}(A, \Gamma^1_n B)$

with $\Gamma^1_n B = B \otimes \mathbb{Z}/2$ for $n \geq 3$ and with $\Gamma^1_2 B = \Gamma B$. Here Γ is the universal quadratic functor of J.H.C. Whitehead [24]. The extension (1) is an easy consequence of the universal coefficient theorem for homotopy groups with coefficients, see for example [13]. In fact, the extension is not split for all $n \geq 2$. For $n \geq 3$ the extension for the category of $\mathbb{Z}/4$-modules in (3.1) is actually a subextension of (1). Again, by (2.3) the extension (1) represents a nontrivial element of the abelian group

(3) $\qquad H^2(\mathbf{Ab}_0, \mathbf{E^n})$ which is $\mathbb{Z}/2$ for $n \geq 3$.

The first named author computed a representing cocycle for the extension in (1), $n \geq 2$. The cohomology groups (3.1) (5) and (3) ($n \geq 3$) are computed in [26].

(3.3) *The category of cyclic groups and the homotopy category of pseudoprojective planes*

We define the category $\mathbf{R}$ of group rings of cyclic groups. Objects are the natural numbers in $\mathbb{N}$. For $f, g \in \mathbb{N}$ the set of morphisms $f \rightarrow g$ is the set

(1) $\qquad \mathbf{R}(f,g) = \{\lambda \in \mathbb{Z}[\mathbb{Z}/g] : f \text{ divides } g \cdot \varepsilon(\lambda)\}$

where ε is the augmentation of the group ring $\mathbb{Z}[\mathbb{Z}/g]$. for $\lambda \in \mathbf{R}(f,g)$ we define the homomorphism

(2) $\qquad \theta_\lambda : \mathbb{Z}/f \rightarrow \mathbb{Z}/g, \qquad \theta_\lambda(1) = g \cdot \varepsilon(\lambda)/f \bmod g.$

Now the composition $\lambda\mu : h \rightarrow f \rightarrow g$ in $\mathbf{R}$ is

(3) $\qquad \lambda\mu = \lambda \cdot \theta_{\lambda *}(\mu) \in \mathbb{Z}[\mathbb{Z}/g].$

Here multiplication is taken in the group ring. It is easy to check that $\mathbf{R}$ is a well-defined category.

We define two natural relations $\simeq$ and $\approx$ on $\mathbf{R}$ as follows:

$$(4) \qquad \lambda \simeq \mu \Leftrightarrow \begin{cases} \theta_\lambda = \theta_\mu \text{ and } \exists \beta \in \mathbb{Z}[\mathbb{Z}/g] \\ \text{with } \lambda - \mu = \theta_{\lambda *}(\partial_f) \cdot \beta. \end{cases}$$

Here ∂_f is the sum of all generators in $\mathbb{Z}[\mathbb{Z}/f]$.

$$(5) \qquad \lambda \approx \mu \Leftrightarrow \begin{cases} \theta_\lambda = \theta_\mu \text{ and} \\ \lambda - \mu = (\lambda - \mu \cdot \theta_{\lambda *}(1). \end{cases}$$

We have the canonical linear extension of categories

$$(6) \qquad \bar{\bar{E}} + \to \mathbf{R}/\simeq \xrightarrow{p} \mathbf{R}/\approx$$

Here $\bar{\bar{E}}$ is the natural system defined for $\{\lambda\}$ in $\mathbf{R}/\approx$ by the group cohomology

$$(7) \qquad \bar{\bar{E}}_{\{\lambda\}} = H^2(\mathbb{Z}/f, \theta_\lambda^* I_g), \qquad \lambda : f \to g,$$

where I_g denotes the augmentation ideal of $\mathbb{Z}[\mathbb{Z}/g]$. If l is the order of kernel(θ_λ), then the group in (7) is isomorphic to

$$(8) \qquad \bar{\bar{E}}_{\{\lambda\}} \cong \begin{cases} 0 & \text{if } \theta_\lambda \text{ is injective or surjective,} \\ (\mathbb{Z}/l)^{(g \cdot l/f) - 1} & \text{otherwise.} \end{cases}$$

This shows that the natural system $\bar{\bar{E}}$ is *not* a bimodule. It is an open problem whether the extension (6) is split or not and what the universal graded group $H^*(\mathbf{R}/\approx, \bar{\bar{E}})$ could be.

The extension in (6) has a nice *topological interpretation*: for each $f \in \mathbb{N}$ let

$$(9) \qquad P_f = S^1 \cup_f e^2$$

be the pseudoprojective plane with $\pi_1(P_f) = \mathbb{Z}/f$. There is a canonical isomorphism of categories

$$(10) \qquad r : \mathbf{R}/\simeq \, \cong \mathbf{P}$$

where $\mathbf{P}$ is the full homotopy category of base point preserving maps between pseudoprojective planes. Moreover the equivalence relation $\approx$ in (5) satisfies

$$(11) \qquad \lambda \approx \mu \Leftrightarrow \begin{cases} \text{the maps } r\{\lambda\}, r\{\mu\} : P_f \to P_g \text{ induce the same homomor-} \\ \text{phism on the homotopy groups } \pi_1 \text{ and } \pi_2. \end{cases}$$

The groups of automorphisms in the extension (6) were considered by Olum [19] who also showed that the structure of these groups is surprisingly rich and that this structure is related to rather deep results and problems in algebraic number theory. The authors expect that the same is true for the extension in (6) and for the universal cohomology groups $H^*(\mathbf{R}/\approx, \bar{\bar{E}})$. The results (6) and (10) are proved in [1].

(3.4) *The homotopty category of Eilenberg–MacLane fibrations*

Let π be a group and let $\mathbf{M}_\pi$ be the small category of finitely generated right π-modules. We define for each $n \geq 2$ the *category of k-invariants* $\mathbf{k}_\pi^{n+1}$: Objects are pairs (A, k) where A is an object in $\mathbf{M}_\pi$ and k is an element

(1) $k \in H^{n+1}(\pi, A)$,

morphisms $f:(A',k') \to (A,k)$ are π-linear maps $f: A' \to A$ which satisfy $f_*(k') = k$. We choose for each object (A,k) a fibration E_k over the Eilenberg-MacLane space $K(\pi,1)$:

(2) $K(A,n) \hookrightarrow E_k \twoheadrightarrow K(\pi,1)$

with fibre $K(A,n)$ which is determined up to equivalence by the k-invariant k, compare [3]. Now we have the linear extension of categories

(3) $H^n + \to \mathbf{K}_\pi^n \xrightarrow{\pi_n} \mathbf{k}_\pi^{n+1}$

where the category $\mathbf{K}_\pi^n$ is the full homotopy category of maps over $K(\pi,1)$ consisting of the fibrations E_k in (2). The functor π_n is given by the n-th homotopy group. Moreover, the natural system H^n is the module

(4) $H^n : \mathbf{k}_\pi^{n+1} \subset \mathbf{M}_\pi \xrightarrow{H^n(\pi,\,\cdot)} \mathbf{Ab}$

which carries the object (A,k) to the cohomology $H^n(\pi, A)$ of the group π. It is not known to us whether the extension (3) splits, but probably it does not. On the full subcategory of objects (A,k) with $k=0$, however, the extension splits. It is an interesting fact that the cohomology groups

(5) $H^*(\mathbf{k}_\pi^{n+1}, H^n), \quad n \geq 2,$

are new invariants of the group π. This example is discussed in [2].

4. Homological algebra in functor categories

We first recall from the literature some facts on functor categories. For a small category $\mathbf{C}$ let $\mathbf{Fun(C, Ab)}$ be the category of functors from $\mathbf{C}$ to the category of abelian groups $\mathbf{Ab}$. Morphisms are the natural transformations.

(4.1) **Remark.** $\mathbf{Fun(C, Ab)}$ is an abelian category with enough projectives and by a theorem of Grothendieck [11] also with enough injectives. Compare [8].

We denote by $\mathrm{Hom}_{\mathbf{C}}(F, G)$ the abelian group of all natural transformations $F \to G$ in $\mathbf{Fun(C, Ab)}$. By (4.1) each F has a projective resolution

$$P_*: \quad \cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to F \to 0.$$

On the other hand by (4.1) also each G has an injective resolution

$$I^*: \quad \cdots \leftarrow I^n \leftarrow I^{n-1} \leftarrow \cdots \leftarrow I^0 \leftarrow G \leftarrow 0.$$

Now the functors $\mathrm{Ext}^n_{\mathbf{C}}(\cdot,\cdot)$ derived from the bifunctor $\mathrm{Hom}_{\mathbf{C}}(\cdot,\cdot)$ are given by the cohomology

(4.2) $\qquad \operatorname{Ext}^n_{\mathbf{C}}(F,G) = H^n \operatorname{Hom}_{\mathbf{C}}(P_*,G) = H^n \operatorname{Hom}_{\mathbf{C}}(F,I^*)$.

It is a wellknown fact of homological algebra that the cohomologies in (4.2) defined in terms of P_* and I^* are naturally isomorphic, see for example [18].

The cohomology $H^n(\mathbf{C},D)$ of a small category $\mathbf{C}$ with coefficients in a natural system D can be described by an Ext^n functor on the category **Fun**(F**C**, **Ab**) where $F\mathbf{C}$ is the category of factorizations, see Section 1. Here we use the canonical functor

(4.3) $\qquad \mathbb{Z}:\mathbf{K}\to\mathbf{Ab}$

which carries an object x of the category $\mathbf{K}$ to the free abelian group with one generator x, denoted by $\mathbb{Z}\{x\}$, and which carries a morphism $m:x\to y$ to the isomorphism $m_*:\mathbb{Z}\{x\}\cong\mathbb{Z}\{y\}$ with $m_*(x)=y$. Clearly $\mathbb{Z}$ is isomorphic to the constant functor on $\mathbf{K}$ with value $\mathbb{Z}$.

(4.4) **Theorem.** *For $\mathbb{Z}:F\mathbf{C}\to\mathbf{Ab}$ there is an isomorphism*

$$H^n(\mathbf{C},D) = \operatorname{Ext}^n_{F\mathbf{C}}(\mathbb{Z},D)$$

which is natural in D.

We proof this result by constructing the *generalized bar resolution* B_* of $\mathbb{Z}$ in **Fun**(F**C**, **Ab**).

Proof of (4.4). For a morphism f in $\mathbf{C}$ let

$$(1)\qquad \begin{aligned} &N_n(f)=\{(\lambda_1,\dots,\lambda_n)\in N_n(\mathbf{C}): f=\lambda_1\circ\cdots\circ\lambda_n\},\\ &N_0(f)=\begin{cases}\emptyset, & f\neq 1,\\ \{1\}, & f=1.\end{cases}\end{aligned}$$

We define a chain complex $B_*=\{B_n,d\}$ of $\mathbb{Z}$ in **Fun**(F**C**, **Ab**) by the functors

$$(2)\qquad \begin{aligned} &B_n: F\mathbf{C}\to\mathbf{Ab},\\ &B_n(f)=\mathbb{Z}N_{n+2}(f), \qquad B_n=0 \quad \text{for } n<-1,\\ &B_n(\alpha,\beta): B_n(f)\to B_n(g),\\ &\quad(\lambda_0,\dots,\lambda_{n+1})\mapsto(\alpha\lambda_0,\lambda_1,\dots,\lambda_n,\lambda_{n+1}\beta).\end{aligned}$$

Here $\mathbb{Z}M$ is the free abelian group generated by the set M. The boundary $d:B_n\to B_{n-1}$ is the natural transformation

$$(3)\qquad \begin{aligned} &d_f: B_n(f)\to B_{n-1}(f),\\ &d_f(\lambda_0,\dots,\lambda_{n+1})=\sum_{k=0}^{n}(-1)^k(\lambda_0,\dots,\lambda_k\lambda_{k+1},\dots,\lambda_{n+1}).\end{aligned}$$

For each f there is a contracting chain homotopy s_f with $d_f s_f + s_f d_f = 1$ on $B_n(f)$ defined by

$$(4)\qquad \begin{aligned} &s_f : B_n(f) \to B_{n+1}(f), \\ &s_f(\lambda_0, \dots, \lambda_{n+1}) = (1, \lambda_0, \dots, \lambda_{n+1}). \end{aligned}$$

The homology of B_* in **Fun**(FC, **Ab**) is given objectwise, that is

$$(5)\qquad H_*(B_*)(f) = \ker d_f / \operatorname{im} d_f.$$

Therefore the homotopy (4) implies $H_*(B_*) = 0$ and thus B_* is exact. We point out that s_f in (4) yields no natural transformation $B_n \to B_{n+1}$ but by (5) this is not needed.

Next consider the definition of F^n in (1.4). There is the natural isomorphism of abelian groups

$$(6)\qquad \operatorname{Hom}_{F\mathbf{C}}(B_n, D) \cong F^n(\mathbf{C}, D), \quad n \geq 0,$$

which carries $F : B_n \to D$ to the function $f \in F^n$ with $f(\lambda_1, \dots, \lambda_n) = F(1, \lambda_1, \dots, \lambda_n, 1)$. Indeed by (6) we have the isomorphism of cochain complexes

$$(7)\qquad \operatorname{Hom}_{F\mathbf{C}}(P_*, D) \cong \{F^n(\mathbf{C}, D), \delta\}$$

where P_* is the part of B_* in nonnegative degrees $n \geq 0$. We have

$$(8)\qquad B_{-1} = \mathbb{Z} : F\mathbf{C} \to \mathbf{Ab}$$

by definition in (2) and in (4.3). Since B_* is exact we see by (8) that P_* is a resolution of $\mathbb{Z}$ in **Fun**(FC, **Ab**). Moreover P_* is projective since by (6) each B_n ($n \geq 0$) is a free natural system. In fact B_n is freely generated by the system of sets $\{N_n(f) : f \in \operatorname{Mor}(\mathbf{C})\}$. Now the isomorphism in (4.4) is induced by the isomorphisms in (6), see (1.4) and (4.2). □

5. Derivations and H^1

A derivation from a group G into a right G-module A is a function $d : G \to A$ with the property

$$(5.1)\qquad d(xy) = (dx)^y + dy.$$

An inner derivation $i : G \to A$ is one for which there exists an element $a \in A$ with $i(x) = a - a^x$. It is a classical result that

$$(5.2)\qquad H^1(G, A) = \operatorname{Der}(G, A)/\operatorname{Ider}(G, A)$$

where Der and Ider denote the abelian groups of derivations and of inner derivations respectively. Compare for example [14].

We now consider derivations from a small category **C** into a natural system D on **C** and show that the cohomology $H^1(\mathbf{C}, D)$ can be described similarly as in (5.2).

In the following definition we use the groups $F^n(\mathbf{C}, D)$ defined in (1.4).

(5.3) **Definition.** A *derivation* $d:\mathbf{C}\to D$ is a function in $F^1(\mathbf{C}, D)$ with

$$d(xy) = x_*(dy) + y^*(dx).$$

An *inner derivation* $i:\mathbf{C}\to D$ is one for which there exists an element $a\in F^0(\mathbf{C}, D)$ such that for $x: A\to B$

$$d(x) = x_*a(A) - x^*a(B).$$

(5.4) **Example.** Let $\mathbf{G}$ and D^A be defined as in (1.20). Then a derivation $\mathbf{G}\to D^A$ is exactly given by a derivation $G\to A$. The same holds for inner derivations.

We denote by $\mathrm{Der}(\mathbf{C}, D)$ and $\mathrm{Ider}(\mathbf{C}, D)$ the abelian groups of all derivations and of all inner derivations $\mathbf{C}\to D$ respectively. These are actually functors

(5.5) $\qquad \mathrm{Der}, \mathrm{Ider}: \mathbf{Nat}\to\mathbf{Ab}$

which are defined on morphisms $(\phi^{\mathrm{op}}, \tau)$ exactly as in (1.10).

(5.6) **Remark.** There is a natural isomorphism

$$H^1(\mathbf{C}, D)\cong \mathrm{Der}(\mathbf{C}, D)/\mathrm{Ider}(\mathbf{C}, D)$$

of functors on **Nat**.

This is clear since derivations are just cocycles and inner derivations are just coboundaries in the cochain complex $\{F^*(\mathbf{C}, D), \delta\}$, see (1.4). For later use we consider the *augmentation ideal* $J(\mathbf{C})\in \mathbf{Fun}(F\mathbf{C}, \mathbf{Ab})$. This is the natural system determined by the exact sequence

(5.7) $\qquad 0\to J(\mathbf{C})\to B_0(\mathbf{C})\xrightarrow{d}\mathbb{Z}\to 0$

where $B_0(\mathbf{C})$ and d are the same as in the proof of (4.4). We point out that the kernel $J(\mathbf{C})$ of d is given objectwise, that is $J(\mathbf{C})(f) = \mathrm{kernel}\, d_f$.

(5.8) **Lemma.** *There is a natural isomorphism*

$$j^*: \mathrm{Hom}_{F\mathbf{C}}(J(\mathbf{C}), D)\cong \mathrm{Der}(\mathbf{C}, D)$$

of functors on **Nat**.

Proof. We have a derivation

(1) $\qquad j:\mathbf{C}\to J(\mathbf{C}), \qquad j(f) = (1, f) - (f, 1)$

and we set $j^*(t) = t\circ j$. We define the inverse k of j^* by the formula

(2) $$k(d)(x)=\sum_i n_i \alpha_{i*} d(\beta_i) \in D_f$$

for $x=\sum_i n_i(\alpha_i, \beta_i) \in J(\mathbf{C})(f)$ and $d \in \mathrm{Der}(\mathbf{C}, D)$. □

6. Cohomological dimension one

The cohomological dimension of a small category **C** and of a group G are fundamental notions in the literature which have been discussed by many authors. We introduce five possibly different dimensions of **C** depending on the type of coefficients, see (1.18).

(6.1) **Definition.** Let $\dim(\mathbf{C}) \leq \infty$ be defined by the condition that $\dim(\mathbf{C}) \leq N$ if $H^n(\mathbf{C}, D)=0$ for $n>N$ and for all natural systems D. We define the dimensions

$$\mathrm{dbim}(\mathbf{C}), \quad \mathrm{dmod}(\mathbf{C}), \quad \mathrm{dloc}(\mathbf{C}) \quad \text{and} \quad \mathrm{dtriv}(\mathbf{C})$$

in the same way where, however, D ranges over all **C**-bimodules, **C**-modules, local systems on **C** and trivial systems on **C** respectively, see (1.18).

Clearly by the definition of the cohomology groups in (1.18) we have the inequalities

(6.2) $$\mathrm{dtriv}(\mathbf{C}) \leq \mathrm{dloc}(\mathbf{C}) \leq \mathrm{dmod}(\mathbf{C}) \leq \mathrm{dbim}(\mathbf{C}) \leq \dim(\mathbf{C}).$$

(6.3) **Theorem.** (A) *If* **F** *is a free category, then* $\dim(\mathbf{F}) \leq 1$.

(B) *If* **C** *is a small category and if* $\Sigma^{-1}\mathbf{C}$ *is the localization of* **C** *with respect to a subset* Σ *of* $\mathrm{Mor}(\mathbf{C})$, *then*

$$\dim(\mathbf{C}) \leq 1 \Rightarrow \dim(\Sigma^{-1}\mathbf{C}) \leq 1.$$

(6.4) **Remark.** The theorem generalizes the following wellknown facts of homological algebra (see for example [6], [14] and [16]):

(A) If F is a *free monoid*, then the associated small category **F** with a single object and with $\mathrm{Mor}(\mathbf{F})=F$ is a free category and thus

$$\mathrm{cd}(F)=\mathrm{dmod}(\mathbf{F}) \leq \dim(\mathbf{F}) \leq 1$$

by (6.3)(A) and (6.2). Here $\mathrm{cd}(F)$ is the usual cohomological dimension with respect to coefficients in left F-modules.

(B) If G is the free group generated by a set S, then we have

$$\mathbf{G}=S^{-1}\mathbf{F}$$

where $F=\mathrm{Mon}(S)$ is the free monoid generated by S. Now (6.3)(B) shows

$$\mathrm{cd}(G)=\mathrm{dmod}(\mathbf{G}) \leq \dim(S^{-1}\mathbf{F}) \leq 1.$$

Here again $\mathrm{cd}(G)$ is the usual cohomological dimension of G.

(6.5) **Remark.** Moreover Theorem (6.3) corresponds to the following results of Cheng–Wu–Mitchell [7]:

(A) $\mathrm{dbim}(\mathbf{F}) \leq 1$ for a free category $\mathbf{F}$.

(B) $\mathrm{dbim}(\mathbf{C}) \leq 1 \Rightarrow \mathrm{dbim}(\Sigma^{-1}\mathbf{C}) \leq 1$.

This as well generalizes the classical results in (6.4).

(6.6) **Remark.** It is clear that for a free category $\mathbf{F}$ each linear extension of $\mathbf{F}$ is a split extension. Therefore the classification in (2.3) shows that for all natural systems D on $\mathbf{F}$ we have

$$H^2(\mathbf{F}, D) = 0.$$

By (6.3)(A) we actually know that for all $n \geq 2$ also $H^n(\mathbf{F}, D) = 0$.

For the proof of Theorem (6.3) we use the following lemmas.

(6.7) **Lemma.** $\dim(\mathbf{C}) \leq 1 \Leftrightarrow J(\mathbf{C})$ *projective.*

Here $J(\mathbf{C})$ is the augmentation ideal in (5.7).

(6.8) **Lemma.** *The localization* $q : \mathbf{C} \to \Sigma^{-1}\mathbf{C}$ *induces the isomorphism*

$$q^* : \mathrm{Der}(\Sigma^{-1}\mathbf{C}, D) \cong \mathrm{Der}(\mathbf{C}, q^*D)$$

which is natural in $D \in \mathbf{Fun}(F(\Sigma^{-1}\mathbf{C}), \mathbf{Ab})$, *compare* (5.5).

(6.9) **Proof of Theorem (6.3).** Proposition (A) is equivalent to the following statements (1), (2), (3) which are all equivalent to each other:

(1) $J(\mathbf{F})$ is projective, see (6.7).

(2) $\mathrm{Hom}_{F\mathbf{F}}(J(\mathbf{F}), \cdot)$ is an exact functor.

(3) $\mathrm{Der}(\mathbf{F}, \cdot)$ is an exact functor, see (5.8).

Here (3) follows from the presentation

$$(4) \qquad \mathrm{Der}(\mathbf{F}, D) = \prod_{f \in \mathbf{S}} D_f$$

where $\mathbf{F}$ is freely generated by $\mathbf{S} \subset \mathrm{Mor}(\mathbf{F})$.

Similarly proposition (B) is a consequence of (6.8) and (6.7) since (1), (2) and (3) are as well equivalent for $\mathbf{F}$ replaced by $\Sigma^{-1}\mathbf{C}$ and $\mathbf{C}$ respectively. □

(6.10) **Proof of (6.7).** '$\Rightarrow$' is obvious.

Now assume $\dim(\mathbf{C}) \leq 1$. For the exact sequence (5.9) we have the short exact sequence

(1) $$0 \to K \to B_1 \xrightarrow{d} J \to 0$$

which induces the exact sequence

(2) $$H^1(\mathbf{C}, B_1) \xrightarrow{d_*} H^1(\mathbf{C}, J) \xrightarrow{\delta} H^2(\mathbf{C}, K) = 0$$

where δ is the Bockstein homomorphism. Here $H^2(\mathbf{C}, K) = 0$ since $\dim(\mathbf{C}) \le 1$. Now d_* is embedded in the commutative diagram with short exact rows, see (5.6):

(3) $$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ider}(\mathbf{C}, B_1) & \to & \mathrm{Der}(\mathbf{C}, B_1) & \to & H_1(\mathbf{C}, B_1) & \to & 0 \\ & & \downarrow d_1 & & \downarrow d_2 & & \downarrow d_* & & \\ 0 & \to & \mathrm{Ider}(\mathbf{C}, J) & \to & \mathrm{Der}(\mathbf{C}, J) & \to & H^1(\mathbf{C}, J) & \to & 0 \end{array}$$

Here d_1 and d_2 are as well induced by d in (1). Since d_* is surjective by (2) and since d_1 is seen to be surjective by use of the definition of inner derivations in (5.3) also d_2 is surjective. Now by (5.8) the homomorphism d_2 is isomorphic to

(4) $$d_* : \mathrm{Hom}_{F\mathbf{C}}(J, B_1) \to \mathrm{Hom}_{F\mathbf{C}}(J, J).$$

Therefore there is $s: J \to B_1$ in $\mathbf{Fun}(F\mathbf{C}, \mathbf{Ab})$ with $ds = 1$. Since B_1 is free J is projective. □

(6.11) **Proof of (6.8).** We construct the inverse k of q^* in (6.8) as follows: Choose a free category $\mathbf{F(S)}$ generated freely by $\mathbf{S}$ such that $\mathbf{C}$ is a quotient

(1) $$p : \mathbf{F(S)} \to \mathbf{C} = \mathbf{F(S)}/\sim .$$

This yields the quotient functor

(2) $$p' : \mathbf{F}(\mathbf{S} \dot{\cup} \Sigma^{\mathrm{op}}) \to \Sigma^{-1}\mathbf{C} = \mathbf{F}(\mathbf{S} \dot{\cup} \Sigma^{\mathrm{op}})/\sim'.$$

Here $\sim'$ is the natural equivalence relation generated by $\sim$ on $\mathbf{F(S)}$ and by

(3) $$\bar{\sigma} \cdot \sigma^{\mathrm{op}} \sim' 1, \qquad \sigma^{\mathrm{op}} \cdot \bar{\sigma} \sim' 1 \quad \text{for } \sigma \in \Sigma$$

where $\bar{\sigma} \in \mathbf{F(S)}$ represents σ.

For a derivation $d : \mathbf{C} \to q^*D$ we define the derivation

(4) $$\begin{gathered} \bar{k}(d) : \mathbf{F}(\mathbf{S} \dot{\cup} \Sigma^{\mathrm{op}}) \to D \\ \text{with } \bar{k}(d)(s) = d(ps) \in D_{q(ps)} \quad \text{for } s \in \mathbf{S}, \\ \bar{k}(D)(\sigma^{\mathrm{op}}) = -(q\sigma)_*^{-1}((q\sigma)^*)^{-1} d\sigma \quad \text{for } \sigma \in \Sigma. \end{gathered}$$

The second equation corresponds to the fact that for a derivation d and for $1 = ee^{-1}$ we have

(5) $$0 = d(1) = d(ee^{-1}) = e_* d(e^{-1}) + (e^{-1})^* d(e).$$

This shows as well that $\bar{k}(d)$ factors over p' in (2). The factorization $k(d)$ with

$\bar{k}(d) = k(d)p'$ yields the homomorphism $k: d \mapsto k(d)$ which is the inverse of q^*. □

7. Computations of the cohomology by covers

We use a method which is analogous to the computation of the Čech cohomology of a space by covers, see for example [4].

A *double cochain complex* $K = (K^{p,q}, d, \delta)$, $p, q \in \mathbb{Z}$, consists of abelian groups $K^{p,q}$ and homomorphisms

$$d = d^{p,q} : K^{p,q} \to K^{p+1,q}, \qquad \delta = \delta^{p,q} : K^{p,q} \to K^{p,q+1} \tag{7.1}$$

such that $\delta\delta = 0$, $dd = 0$, and $\delta d = d\delta$.

K is called *(M, N)-acyclic* if

$$H^n(K^{*,q}, d) = 0 \quad \text{for } n > M,\ q \geq N, \qquad H^n(K^{p,*}, \delta) = 0 \quad \text{for } n > N,\ p \geq M. \tag{7.2}$$

The following proposition is a standard fact proved by the spectral sequence which computes the cohomology of the total complex.

(7.3) **Proposition.** *Suppose K is (M, N)-acyclic. Then*

$$H^{N+n}(S_K^*) = H^{M+n}(W_K^*), \quad n \in \mathbb{Z},$$

where S_K^* and W_K^* are the cochain complexes defined by

$$S_K^* = (\text{kernel}(d^{N,*}), \delta) \quad \textit{for } * \geq M \textit{ and } S_K^* = 0 \textit{ otherwise},$$

$$W_K^* = (\text{kernel}(\delta^{*,M}), d) \quad \textit{for } * \geq N \textit{ and } W_K^* = 0 \textit{ otherwise}.$$

For a small category $\mathbf{C}$ and for a sequence $U_\alpha, \alpha \in \mathbb{N}$, of subcategories of $\mathbf{C}$ we introduce a double complex K as follows: Let $U_{\alpha_0 \cdots \alpha_p} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ be the intersection category and let

$$K^{p,q} = \underset{\alpha_0 < \cdots < \alpha_p}{\times} F^q(U_{\alpha_0 \cdots \alpha_p}, D), \tag{7.4}$$

see (1.4). Here D is a natural system on $\mathbf{C}$ which yields by restriction a natural system on each subcategory of $\mathbf{C}$. The coboundary $\delta^{p,q}$ is given by the direct product of the corresponding coboundary δ in (1.4). The coboundary d carries a sequence $\omega = (\omega_{\alpha_0 \cdots \alpha_p} : \alpha_0 < \cdots < \alpha_p)$ to the sequence $d\omega = ((d\omega)_{\alpha_0 \cdots \alpha_{p+1}} : \alpha_0 < \cdots < \alpha_{p+1})$ where

$$(d\omega)_{\alpha_0 \cdots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \cdots \alpha_{i-1}\alpha_{i+1} \cdots \alpha_{p+1}}. \tag{7.5}$$

The terms in the sum are restricted to the subcategory $U_{\alpha_0 \cdots \alpha_{p+1}}$ and the sum is taken in the group $F^n(U_{\alpha_0 \cdots \alpha_{p+1}}, D)$. Now the crucial lemma is:

(7.6) **Lemma.** *K in (7.4) is a $(0,\infty)$-acyclic double complex. Moreover, if $\dim(U_{\alpha_0 \cdots \alpha_p}) \le l$, see (6.1), then K is $(0,l)$-acyclic.*

For the proof see [25].

The lemma implies by (7.3):

(7.7) **Proposition.** *Suppose the sequence (U_α) has the following properties* (a) *and* (b):

(a) $\dim(U_{\alpha_0 \cdots \alpha_p}) \le 1$ *for all* $\alpha_0 < \cdots < \alpha_p$,

(b) $N_b(\mathbf{C}) = \bigcup_{\alpha \in \mathbb{N}} N_b(U_\alpha)$, *see* (1.4).

Then $H^{1+n}(\mathbf{C}, D) = H^{1+n}(S_K^*) = H^n(W_K^*)$ for all n with $1 < 1+n < b$.

This proposition can be used for effective computation of the cohomology $H^{1+n}(\mathbf{C}, D)$. Condition (a), in particular, is satisfied if all intersections $U_{\alpha_0 \cdots \alpha_p}$ are free categories, see Theorem (6.3).

We describe a simple example for the computation of H^2, where we as well use the result (5.6). Consider the commutative square

(7.8)
$$\mathbf{Q} = \left\{ \begin{array}{ccc} & b & \\ {}^{\alpha}\nearrow & & \searrow^{\beta} \\ a & \xrightarrow[\varepsilon]{} & d, \quad \beta\alpha = \varepsilon = \delta\gamma \\ {}_{\gamma}\searrow & & \nearrow_{\delta} \\ & c & \end{array} \right\}.$$

For this category we have the free subcategories $U_1 = \{\alpha, \beta\}$, $U_2 = \{\gamma, \delta\}$ and $U_{12} = \{\varepsilon\}$. The sequence (U_1, U_2) satisfies the conditions (a) nad (b) in Proposition (7.7) for any b. Therefore $H^2(\mathbf{Q}, D) = H^1(W_K^*)$ where D is a natural system on $\mathbf{Q}$. The first part of W_K^* is

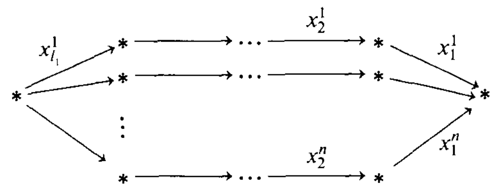
(7.9)
$$W_K^0 = D_\alpha \times D_\beta \times D_\gamma \times D_\delta \xrightarrow{d} W_K^1 = D_\varepsilon \to W_K^2 = 0$$

with $d(\omega_\alpha, \omega_\beta, \omega_\gamma, \omega_\delta) = \alpha^*\omega_\beta + \beta_*\omega_\alpha - \delta_*\omega_\gamma - \gamma^*\omega_\delta$. It follows that

(7.10)
$$H^2(\mathbf{Q}, D) = D_\varepsilon / (\alpha^* D_\beta + \beta_* D_\alpha + \delta_* D_\gamma + \gamma^* D_\delta).$$

We remark that by (2.3) each cohomology class in this group represents a category which is a linear extension of the commutative square.

In a similar way as (7.10) one computes the cohomology of the category $\mathbf{K}$ given by the following finite commutative diagram

(7.11) 

Here all places $*$ correspond to pairwise different objects. By commutativity we have

$$x = x_1^k \circ x_2^k \circ \cdots \circ x_{l_k}^k \quad \text{for } k = 1, \ldots, n.$$

For any natural system D on $\mathbf{K}$ the second cohomology is

$$(7.12) \qquad H^2(\mathbf{K}, D) = \mathop{\times}_{k=2}^{n} (D(x)/I_k)$$

with

$$I_k = \sum_{i=1}^{l_1} (x_1^1)_* \cdots (x_{i-1}^1)_* (x_{l_1}^1)^* \cdots (x_{i+1}^1)^* D(x_i^1)$$
$$+ \sum_{j=1}^{l_k} (x_1^k)_* \cdots (x_{j-1}^k)_* (x_{l_k}^k)^* \cdots (x_{j+1}^k)^* D(x_j^k).$$

In particular, if $n = 1$ the term (7.12) is trivial, indeed, this is clear since $\mathbf{K}$ is a free category in this case, see (6.3).

8. Further cohomologies of small categories

In this section we consider again the various cohomologies defined in (1.18) and compare them with the corresponding notions in the literature. This is the program announced in Remark (1.19). Recall from Section 4 that homological algebra (with derived functors) is available in any functor category $\mathbf{Fun}(\mathbf{K}, \mathbf{Ab})$.

(8.1) **Definition.** Let $\mathbf{C}$ be a small category, and let M be a $\mathbf{C}$-bimodule.
We have the canonical $\mathbf{C}$-bimodule

(1) $\qquad \mathbb{Z}\mathbf{C} : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Ab},$

which carries the object (A, B) to the free abelian group $\mathbb{Z}\mathbf{C}(A, B)$ generated by the morphism set $\mathbf{C}(A, B)$. The *Hochschild-Mitchell* cohomology of $\mathbf{C}$ with coefficients in M is defined by

(2) $\qquad \mathrm{Ext}^n_{\mathbf{C}^{\mathrm{op}} \times \mathbf{C}}(\mathbb{Z}\mathbf{C}, M)$

as described in (4.2), see [17].

(8.2) **Definition.** Let $\mathbf{C}$ be a small category and F be a $\mathbf{C}$-module. For the constant functor

(1) $\mathbb{Z}: \mathbf{C} \to \mathbf{Ab}$,

compare (4.3), we have by (4.2) as well the groups

(2) $\mathrm{Ext}^n_{\mathbf{C}}(\mathbb{Z}, F)$.

This is a description of the cohomology of a small category used by Watts [23] and Quillen [21]. Moreover, it is particular case of the cohomology of topoi in the sense of Grothendieck [15].

(8.3) **Remark.** Consider the Grothendieck topos

(1) $\hat{\mathbf{C}} = \mathbf{Fun}(\mathbf{C}, \mathbf{Set})$

where **Set** is the category of sets. The abelian group objects in $\hat{\mathbf{C}}$ are just the **C**-modules. Now the Grothendieck cohomology of $\hat{\mathbf{C}}$ with coefficients in the abelian group object F is exactly the group (8.2)(2) since $\mathbb{Z}$ is the free abelian group object over the terminal object of $\hat{\mathbf{C}}$, compare [15]. In Section 29 of [12] Grothendieck calls a functor $\phi: \mathbf{C} \to \mathbf{C}'$ of small categories a *weak equivalence* if ϕ induces an isomorphism

(2) $\phi^*: H^n(\mathbf{C}', F) \overset{\cong}{\to} H^n(\mathbf{C}, \phi^* F)$

for all n and all $\mathbf{C}'$-modules F and if ϕ satisfies in degrees $n=0$, $n=1$ certain additional nonabelian criteria. It is as well justified to define notions of weak equivalences by using in (2) bimodules or even natural systems as coefficients. We do not know whether these notions are actually stronger than the notion of Grothendieck.

(8.4) **Remark.** We know by (4.2) that the group $\mathrm{Ext}^n_{\mathbf{C}}(\mathbb{Z}, F)$ can be defined by a projective resolution B_* of $\mathbb{Z}$ or by an injective resolution I^* of F in the category of **C**-modules. When we take the injective resolution we obtain canonically the derived of the functor $L = \varprojlim$ by

(1) $\mathrm{Ext}^n_{\mathbf{C}}(\mathbb{Z}, F) = H^n \, \mathrm{Hom}_{\mathbf{C}}(\mathbb{Z}, I^*) = H^n(L(I^*)) = \varprojlim^{(n)}(F)$.

since $\varprojlim(\cdot) = \mathrm{Hom}_{\mathbf{C}}(\mathbb{Z}, \cdot)$. When we take the projective resolution B_* of $\mathbb{Z}$ we obain

(2) $\mathrm{Ext}^n_{\mathbf{C}}(\mathbb{Z}, F) = H^n \, \mathrm{Hom}_{\mathbf{C}}(B_*, F)$.

The cochain complex $\mathrm{Hom}_{\mathbf{C}}(B_*, F)$ is isomorphic to the cochain complex constructed by Roos [22] and more generally by Bousfield–Kan [5]. Therefore the equation

(3) $H^n \, \mathrm{Hom}_{\mathbf{C}}(B_*, F) = \varprojlim^{(n)}(F)$

corresponds to the classical result of Roos in the case of **C**-modules. This actually is a very special case of the result of Roos and Bousfield–Kan since they consider

$\varprojlim^{(n)}(F)$ for functors $F: \mathbf{C} \to \mathbf{A}$ where $\mathbf{A}$ is a suitable abelian category. Here a representation like $\varprojlim(\cdot) = \mathrm{Hom}_{\mathbf{C}}(\mathbb{Z}, \cdot)$ is not available and therefore a simple argument as in (1) does not work.

(8.5) **Proposition.** (A) *For any* $\mathbf{C}$*-bimodule* M *we have*

$$H^n(\mathbf{C}, M) = \mathrm{Ext}^n_{F\mathbf{C}}(\mathbb{Z}, \pi^* M) = \mathrm{Ext}^n_{\mathbf{C}^{\mathrm{op}} \times \mathbf{C}}(\mathbb{Z}\mathbf{C}, M).$$

(B) *For any* $\mathbf{C}$*-module* F *we have*

$$H^n(\mathbf{C}, F) = \mathrm{Ext}^n_{F\mathbf{C}}(\mathbb{Z}, \pi^* p^* F) = \mathrm{Ext}^n_{\mathbf{C}}(\mathbb{Z}, F) = \varprojlim^{(n)}(F).$$

Proof. (A) Consider the standard projective bar resolution of Mitchell in [17, p. 70f]. If we compute the term on the right hand side of (A) we get our definition of $H^n(\mathbf{C}, M)$, compare (1.4) and (1.18)(1).

(B) Consider the projective resolution B_* of $\mathbb{Z}$ in (8.4). Now in the same way the right hand side of (B) canonically coincides with our definition of $H^n(\mathbf{C}, F)$, compare (1.4) and (1.18)(2). □

(8.6) **Remark.** Quillen proves in [21, p. 91], that for a local system L on $\mathbf{C}$ one has the natural equation

$$\varprojlim^{(n)}(q^* L) = H^n(B\mathbf{C}, L).$$

Here $q: \mathbf{C} \to \pi\mathbf{C}$ is the localization functor in (1.16) and $B\mathbf{C}$ is the classifying space of the category $\mathbf{C}$ (the realization of the simplicial set defined by the *nerve* of $\mathbf{C}$, see also (1.4)). The fundamental groupoid of the space $B\mathbf{C}$ is canonically equivalent to the fundamental groupoid $\pi\mathbf{C}$. Therefore L determines a local system of coefficients for the space $B\mathbf{C}$ which we as well denote by L.

(8.7) **Remark.** By (8.5)(B) we know

(1) $\qquad H^n(\mathbf{C}, D) = H^n(F\mathbf{C}, D)$

where the left hand side is a cohomology with coefficients in a natural system D. The right hand side is the cohomology with coefficients in the $F\mathbf{C}$-module D which as we know by (1.18) is a special case of the cohomology of $F\mathbf{C}$ with coefficients in a natural system on $F\mathbf{C}$. Therefore one might consider inductively the cohomology groups

(2) $\qquad H^n(F^i\mathbf{C}, D), \quad i \geq 1,$

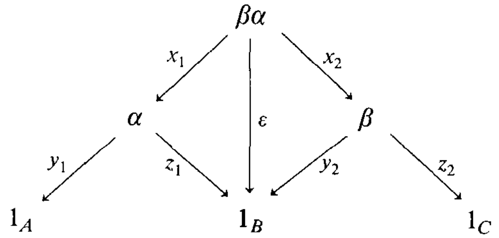
with $D \in \mathbf{Fun}(F^i\mathbf{C}, \mathbf{Ab})$ as being a generalized cohomology of $\mathbf{C}$. Any reasonable cohomology of $\mathbf{C}$, however, should have the property that the cohomology vanishes in degree ≥ 2 if $\mathbf{C}$ is free. This property is satisfied for the group (2) if $i = 1$, see (6.3)(A), yet the example below shows that this property does not hold if $i > 1$.

(8.8) **Example.** Consider the free category $\mathbf{C}$ generated by the graph

(1) $A \xrightarrow{\alpha} B \xrightarrow{\beta} C.$

Then FC is the category pictured in the commutative diagram

(2)



In particular, FC is not a free category though $\mathbf{C}$ is free. In addition, we show

(3) $\operatorname{dmod}(F\mathbf{C}) = \dim(\mathbf{C}) \leq 1 < 2 = \dim(F\mathbf{C}),$

compare the notation in (6.1). The first part is clear by (8.7)(1) and (6.3)(A). Moreover, the last equation follows by the computation of cohomology groups $H^n(F\mathbf{C}, D)$, D a natural system on $F\mathbf{C}$. Here we use the method of Section 7 which shows similarly as in (7.9) that

(4) $H^2(F\mathbf{C}, D) = D_\varepsilon/(x_1^* D_{z_1} + x_2^* D_{y_2} + z_{1*} D_{x_1} + y_{2*} D_{x_2})$

and $H^n(F\mathbf{C}, D) = 0$ for $n \geq 3$ and any natural system D on $F\mathbf{C}$. Now (4) is non-trivial if we set $D_\varepsilon \neq 0$ and $D_f = 0$ for morphisms f in $F\mathbf{C}$, $f \neq \varepsilon$.

Acknowledgements

The authors would like to acknowledge the support of the Sonderforschungsbereich 40 "Theoretische Mathematik" and the Max-Planck-Institut für Mathematik in Bonn. We also remember with pleasure the stimulating conversations with J. Benabou in Oberwolfach 1983.

Literature

[1] H.-J. Baues, Combinatorial homotopy theory of CW-complexes, Preprint, Max-Planck-Institut für Mathematik, Bonn, 1984.

[2] H.-J. Baues, Homotopical algebra, Preprint, Max-Planck-Institut für Mathematik, Bonn, 1984.

[3] H.-J. Baues, Obstruction Theory, Lecture Notes in Math. 628 (Berlin, Springer, 1977).

[4] R. Bott and L.W. Tu, Differential Forms in Algebraic Topology, GTM 82 (Springer, New York, 1982).

[5] A.K. Bousfield and D.M. Kan, Homotopy Limits, Completions and Localizations, Lecture Notes in Math. 304 (Springer, Berlin, 1972).

[6] H. Cartan and S. Eilenberg, Homological Algebra (Princeton Univ. Press, Princeton, 1956).

[7] Ch.Ch. Cheng, Y.-Ch. Wu and B. Mitchell, Categories of fractions preserve dimension one, Comm. in Algebra 8 (1980) 927–939.

[8] P. Freyd, Abelian Categories (Harper & Row, New York, 1964).
[9] Friedlander, Private communication.
[10] P. Gabriel and M. Zisman, Calculus of Fractions and Homotopy Theory (Springer, Berlin, 1967).
[11] A. Grothendieck, Sur quelque points d'algèbre homologique, Tohoku Math. J. 9 (1957) 119–221.
[12] A. Grothendieck, Persuing Stacks, Preprint, 1983.
[13] P.J. Hilton, Homotopy Theory and Duality (Gordon and Breach, New York, 1965).
[14] P.J. Hilton and U. Stammbach, A Course in Homological Algebra, GTM 4 (Springer, New York, 1971).
[15] P.T. Johnstone, Topos Theory (Academic Press, London, 1977).
[16] S. MacLane, Homology, Grundlehren (Springer, Berlin, 1967).
[17] B. Mitchell, Rings with several objects, Advances in Math. 8 (1972) 1-161.
[18] B. Mitchell, Theory of Categories (Academic Press, New York and London, 1965).
[19] P. Olum, Self-equivalences of pseudoprojective planes, Topology 4 (1965) 109–127.
[20] B. Pareigis, Kategorien und Funktoren (Teubner, Stuttgart, 1969).
[21] D. Quillen, Higher algebraic *K*-theory *I*, in: H. Bass, ed., Algebraic *K*-Theory I, Lectures Notes in Math. 341 (Springer, Berlin, 1973) 85–147.
[22] J.E. Roos, Sur les foncteurs dérivés de $\varprojlim$. Applications, Compte Rendue Acad. Sci. Paris 252 (1961) 3702–3704.
[23] Ch.E. Watts, A Homology Theory for Small Categories, Proc. Conf. on Categorical Algebra, La Jolla, CA, 1965.
[24] J.H.C. Whitehead, A certain exact sequence, Ann. Math. 52 (1950) 51–110.
[25] G. Wirsching, Kohomologie kleiner Kategorien, Dipolomarbeit, Bonn, 1984.
[26] M. Hartl, The second cohomology of the category of finitely generated abelian groups, Preprint and Diplomarbeit, Bonn, 1985.
[27] W.G. Dwyer and D.M. Kan, Function complexes for diagrams of semiplicial sets, Proc. Koninkl. Nederl. Akademie 86(2) (1983) 139–147.